Fibre Product

Note Title 8/19/2019

X, Y, S schemes

Definition: $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ morphisms

The fibre product $X \times_S Y$ is the unique scheme

Comes from the universal property (up to isomorphisms)

s.t. ①
$$\begin{tikzcd} X\times_S Y \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$
commutes

② If
$$\begin{tikzcd} Z \arrow[rr, "g''"] \arrow[dr, "\exists!\theta"] \arrow[dd, "f''"'] & & X \arrow[dd, "f"] \\ & X\times_S Y \arrow[ur, "g'"] \arrow[dl, "f'"] & \\ Y \arrow[rr, "g"'] & & S \end{tikzcd}$$
universal property

pf: (Step 1) $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$ affine

w/ $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ induced by $R \to A$, $R \to B$

then $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$

Indeed, if
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X = \operatorname{Spec} A \arrow[d] \\ Y = \operatorname{Spec} B \arrow[r] & S = \operatorname{Spec} R \end{tikzcd}$$

$\operatorname{Hom}_{Sch}(Z, \operatorname{Spec} A) = \operatorname{Hom}_{Ring}(A, \Gamma(Z, \mathcal{O}_Z))$

From the universal property of the tensor product

$$\approx \quad X\times_S Y = \operatorname{Spec}(A\otimes_R B) \longleftarrow Z$$

$$\begin{tikzcd}
\Gamma(Z,\mathcal{O}_Z) & & \\
& A\otimes_R B \arrow[ul, red, "\exists!\,\theta"'] & A \arrow[l] \arrow[ull, bend right] \\
& B \arrow[u] \arrow[uul, bend left] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

(Step 2) • $\{X_i\}$ cover of X $\Longrightarrow$ $X\times_S Y$ exists

• $X_i\times_S Y$ exists

$$\begin{tikzcd}
X_i\times_S Y \simeq p_1^{-1}(X_i) \arrow[dr] \arrow[rr, bend left] \arrow[d] & & Y \arrow[dd] \\
X_i \arrow[dr, hook, bend right] & X\times_S Y \arrow[ur] \arrow[d, "p_1"] & \\
& X \arrow[r] & S
\end{tikzcd}$$

local model: $X_{ij} \hookrightarrow X_i$

$\Rightarrow$ $X_{ij}\times_S Y \hookrightarrow X_i\times_S Y$

True when $X_{ij} = \operatorname{Spec} A_f$, $X_i = \operatorname{Spec} A$

$A \longrightarrow A_f$

$S = \operatorname{Spec} R$, $Y = \operatorname{Spec} B$

$$A\otimes_R B \longrightarrow A_f\otimes_R B \cong (A\otimes_R B)_{f\otimes 1}$$

$$\begin{tikzcd}
& X_i \\
X_{ij} \arrow[ur] \arrow[dr, hook] & \\
& X_j
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
& X_i\times_S Y \\
X_{ij}\times_S Y \arrow[ur] \arrow[dr, hook] & \\
& X_j\times_S Y
\end{tikzcd}$$

When S is not affine, $S = \bigcup S_i$, $S_i = \operatorname{Spec} R_i$

$X_i \hookrightarrow f^{-1}(S_i)$ affine open subset

then $X\times_S Y = X\times_{S_i} Y$

$$\begin{tikzcd}
X\times_S Y \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X\times_{S_i} Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & S_i \arrow[r] & S
\end{tikzcd}$$

Definition: $f: X \to Y$ morphism of schemes

$y \in Y$, $\quad k(y)$ residue field of y

$\rightsquigarrow \quad X_y := X \times_Y \operatorname{Spec} k(y)$ fibre of f over y

- Can view X as deformation of X_y.
- $$\begin{tikzcd} X \arrow[r] & \operatorname{Spec} \mathbb{Z} \\ X_{(p)} \arrow[u, phantom, "\cup"] \arrow[r] & (p) \arrow[u, phantom, "\in"] \end{tikzcd}$$ defined over $\mathbb{Q}$, then $\underline{X_{(p)}}$ defined over $\mathbb{F}_p$

  reduction modulo p.
- (base change) $$\begin{tikzcd} X' = X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$ ex. moduli problem

base change is transitive:

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

$$(X \times_S S') \times_{S'} S'' \cong X \times_S S''$$

use universal property

ex. $\mathbb{A}^n_k \times_{\operatorname{Spec} k} \mathbb{A}^m_k \cong \mathbb{A}^{n+m}_k$ $\quad$!! $\quad k[x_1, \dots, x_n] \otimes k[y_1, \dots, y_m]$

$$\cong \; k[x_1, \dots, x_n, y_1, \dots, y_m]$$

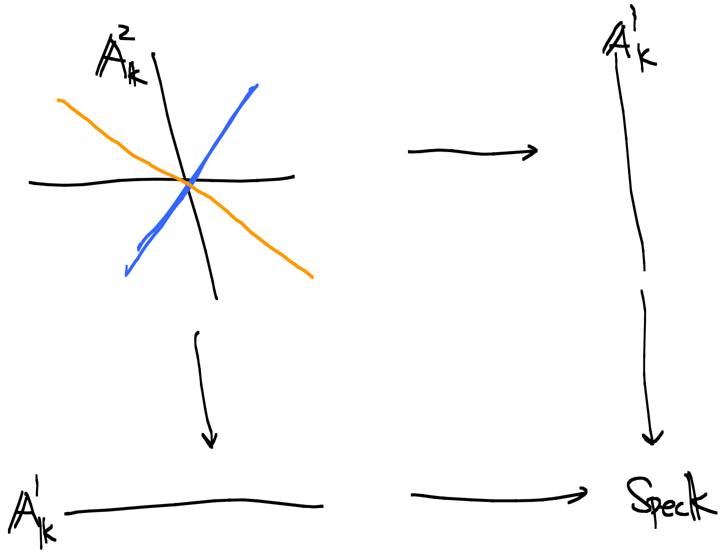
Caveat: Given $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$, one can form

set-theoretic fibre product

$$\underline{X} \times_{\underline{S}} \underline{Y} := \{ (x, y) \in \underline{X} \times \underline{Y} \mid f(x) = g(y) \}$$

However, $\underline{X \times_S Y} \neq \underline{X} \times_{\underline{S}} \underline{Y}$

ex.



$(x-y) \neq (x+y) \in \mathbb{A}^2_k$

thus $\underline{\mathbb{A}^2_k} \neq \underline{\mathbb{A}^1_k} \times \underline{\mathbb{A}^1_k}$